

# THE BERGMAN KERNEL OF A CERTAIN HARTOGS DOMAIN AND THE POLYLOGARITHM FUNCTION

ATSUSHI YAMAMORI

ABSTRACT. We consider a certain Hartogs domain which is related to the Fock-Bargmann space. We give an explicit formula for the Bergman kernel of the domain in terms of the polylogarithm functions. Moreover we solve the Lu Qi-Keng problem of the domain in some cases.

## 1. INTRODUCTION

In this paper we consider a Hartogs domain in  $\mathbb{C}^{n+m}$  defined by the inequality  $\|\zeta\|^2 < e^{-\mu\|z\|^2}$ , where  $(z, \zeta) \in \mathbb{C}^n \times \mathbb{C}^m$  and  $\mu > 0$ . Our aim is to show that the Bergman kernel of this domain can be written explicitly in terms of the polylogarithm functions.

The polylogarithm function appears in many different areas of mathematics. For example it appears in analysis of the Riemann zeta function, algebraic geometry and mathematical physics (cf. [6], [10]). The polylogarithm function is a rational function under a certain condition. Our formula is expressed in terms of these rational cases of the polylogarithm functions and their derivatives.

It is usually hard to obtain an explicit formula of the Bergman kernel of a complex domain. Only few domains with an explicit Bergman kernel are known until now. In this situation, it is fundamental and important to find a domain with explicit Bergman kernel.

There are two basic approaches for obtaining explicit Bergman kernels. One is to construct a complete orthonormal basis of the Bergman space explicitly. For the unit disk, one can find a complete orthonormal basis  $\{\pi^{-1/2}(n+1)^{1/2}z^n\}_{n=1}^\infty$ . However this approach faces difficulty in general. If the domain does not have symmetry, a computation of integral on the domain is unexecutable or extremely difficult.

If the automorphism group of a domain contains enough information (e.g. transitivity) to obtain explicit formula, then we can use it. For example, the Lie group  $SU(n, m)$  acts transitively on the classical domain  $D = \{z \in M_{n,m}(\mathbb{C}); I - \bar{z}^t z > 0\}$  of type I by linear fractional transformation. It is known that the Bergman kernel  $K$  of a classical domain has the property that  $K(z, 0)$  is a non-zero constant. These facts and a transformation rule of the Bergman kernel (cf. [2]) imply that the computation of the Bergman kernel is reduced to the computation of the Jacobian of the linear fractional map. L. K. Hua [11] computed the Bergman kernels for the classical domains in this way. As above, this approach faces difficulty in general.

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The approach in this article is different from the above two. Our method is based on Ligocka's theorem [13] which relates the Bergman kernel of a Hartogs domain to weighted Bergman kernels of the base domain. Thanks to this theorem one can find that our domain and the Fock-Bargmann space are closely related. We will see that Ligocka's theorem and an explicit formula of the Fock-Bargmann kernel lead to an explicit formula of the Bergman kernel of our domain.

As an application of our formula, we solve the Lu Qi-Keng problem for our domain in some cases. The Lu Qi-Keng problem asks whether the Bergman kernel has zeros or not. This problem was investigated for various domains by many authors in this decade. Yin [17] obtained explicit form of the Bergman kernel of the Cartan-Hartogs domain. The Lu Qi-Keng problem for the Cartan-Hartogs domain was studied by several authors (cf. [7]). In [16], the authors obtained an explicit formula of the Bergman kernel of some Hartogs domains and solved the Lu Qi-Keng problem for the domains in some cases. Recently Lu Qi-Keng himself studied the location of the zeros of Bergman kernel in [14]. Further information about the Lu Qi-Keng problem can be found in [2],[3],[12] and [18].

## 2. PRELIMINARIES

Let  $\Omega$  be a domain in  $\mathbb{C}^n$ ,  $L_a^2(\Omega)$  the Hilbert space of square integrable holomorphic functions on  $\Omega$  with the inner product:

$$\langle f, g \rangle = \int_{\Omega} f(z) \overline{g(z)} dz, \quad \text{for all } f, g \in L_a^2(\Omega).$$

The Bergman kernel  $K(z, w) = \overline{K_z(w)}$  is the reproducing kernel for  $L_a^2(\Omega)$ , i.e. if  $f \in L_a^2(\Omega)$  then

$$f(z) = \langle f, K_z \rangle = \int_{\Omega} f(w) K(z, w) dw, \quad \text{for all } z \in \Omega.$$

Let  $\{\phi_k\}$  be a complete orthonormal basis of  $L_a^2(\Omega)$ . Then the Bergman kernel can be also defined by

$$K(z, w) = \sum_k \phi_k(z) \overline{\phi_k(w)}.$$

Let  $p$  be a positive continuous function on  $\Omega$  and  $L_a^2(\Omega, p)$  the Hilbert space of square integrable holomorphic functions with respect to the weight function  $p$  on  $\Omega$  with the inner product

$$\langle f, g \rangle = \int_{\Omega} f(z) \overline{g(z)} p(z) dz, \quad \text{for all } f, g \in L_a^2(\Omega).$$

The weighted Bergman kernel  $K_{\Omega, p}$  of  $\Omega$  with respect to the weight  $p$  is the reproducing kernel of  $L_a^2(\Omega, p)$ .

We define the Hartogs domain  $\Omega_{m, p}$  by

$$\Omega_{m, p} := \{(z, \zeta) \in \Omega \times \mathbb{C}^m; \|\zeta\|^2 < p(z)\}.$$

E. Ligocka [13, Proposition 0] showed that the Bergman kernel of  $\Omega_{m, p}$  is expressed as infinite sum of weighted Bergman kernels of the base domain  $\Omega$ .

**Theorem 1.** *Let  $K_m$  be the Bergman kernel of  $\Omega_{m,p}$  and  $K_{\Omega,p^k}$  the weighted Bergman kernel of  $\Omega$  with respect to the weight function  $p^k$ . Then*

$$K_m((z, \zeta), (z', \zeta')) = \frac{m!}{\pi^m} \sum_{k=0}^{\infty} \frac{(m+1)_k}{k!} K_{\Omega,p^{k+m}}(z, z') \langle \zeta, \zeta' \rangle^k.$$

Here  $(a)_k$  denotes the Pochhammer symbol  $(a)_k = a(a+1) \cdots (a+k-1)$ .

M. Engliš and G. Zhang generalized this theorem for wider class of domains in [8]. Since theorem of this type was first proved by F. Forelli and W. Rudin [9] for  $\Omega$  the unit disk and  $p(z) = 1 - |z|^2$ , some authors call it the Forelli-Rudin construction.

We introduce the polylogarithm function which is necessary to state our main theorem. The polylogarithm function is defined by

$$Li_s(z) = \sum_{k=1}^{\infty} k^{-s} z^k, \quad (1)$$

which converges for  $|z| < 1$  and any  $s \in \mathbb{C}$ . If  $s$  is a negative integer, say  $s = -n$ , then the polylogarithm function has the following closed form:

$$Li_{-n}(z) = \frac{z}{(1-z)^{n+1}} \sum_{j=0}^{n-1} A(n, j+1) z^j$$

where  $A(n, m)$  is the Eulerian number [6, eq.(2.17)]

$$A(n, m) = \sum_{\ell=0}^m (-1)^\ell \binom{n+1}{\ell} (m-\ell)^n.$$

The first few are

$$\begin{aligned} Li_{-1}(z) &= \frac{z}{(1-z)^2}, & Li_{-2}(z) &= \frac{z^2 + z}{(1-z)^3}, \\ Li_{-3}(z) &= \frac{z^3 + 4z^2 + z}{(1-z)^4}, & Li_{-4}(z) &= \frac{z^4 + 11z^3 + 11z^2 + z}{(1-z)^5}. \end{aligned}$$

The polynomial  $A_n(z) = \sum_{j=0}^{n-1} A(n, j+1) z^j$  is called the Eulerian polynomial. More information about the polylogarithm function and the Eulerian polynomial can be found in [4],[6] and [10].

### 3. THE BERGMAN KERNEL OF $D_{n,m}$

Let  $\mu > 0$ . Define  $D_{n,m}$  by

$$D_{n,m} := \{(z, \zeta) \in \mathbb{C}^n \times \mathbb{C}^m; \|\zeta\|^2 < e^{-\mu\|z\|^2}\}.$$

This section is devoted to the study of the Bergman kernel of  $D_{n,m}$ . We shall begin with the Fock-Bargmann space and its reproducing kernel.

The Fock-Bargmann space  $L_a^2(\mathbb{C}^n, e^{-\mu\|z\|^2})$  is the Hilbert space of square integrable entire functions on  $\mathbb{C}^n$  with the inner product

$$\langle f, g \rangle = \int_{\mathbb{C}^n} f(z) \overline{g(z)} e^{-\mu\|z\|^2} dz.$$

The reproducing kernel  $K_{n,\mu}$  of  $L_a^2(\mathbb{C}^n, e^{-\mu\|z\|^2})$  is expressed explicitly as

$$K_{n,\mu}(z, w) = \frac{\mu^n e^{\mu\langle z, w \rangle}}{\pi^n}. \quad (2)$$

The kernel function  $K_{n,\mu}$  is called the Fock-Bargmann kernel (see [1]). We are now ready to state our main result.

**Theorem 2.** *The Bergman kernel of  $D_{n,m}$  is given by*

$$K_{D_{n,m}}((z, \zeta), (z', \zeta')) = \frac{\mu^n}{\pi^{n+m}} e^{\mu \langle z, z' \rangle} \frac{d^m}{dt^m} Li_{-n}(t) \Big|_{t=e^{\mu \langle z, z' \rangle} \langle \zeta, \zeta' \rangle} \quad (3)$$

$$= \frac{\mu^n}{\pi^{n+m}} \frac{d^{m-1}}{dt^{m-1}} \frac{Li_{-(n+1)}(e^{\mu \langle z, z' \rangle} t)}{t} \Big|_{t=\langle \zeta, \zeta' \rangle}. \quad (4)$$

*Proof.* By Ligocka's theorem and the formula (2), we have

$$\begin{aligned} K_{D_{n,m}}((z, \zeta), (z', \zeta')) &= \frac{m!}{\pi^m} \sum_{k=0}^{\infty} \frac{(m+1)_k}{k!} \frac{(k+m)^n \mu^n}{\pi^n} e^{\mu(k+m)\langle z, z' \rangle} \langle \zeta, \zeta' \rangle^k \\ &= \frac{m! \mu^n}{\pi^{n+m}} e^{\mu m \langle z, z' \rangle} \sum_{k=0}^{\infty} \frac{(m+1)_k}{k!} (k+m)^n e^{\mu k \langle z, z' \rangle} \langle \zeta, \zeta' \rangle^k. \end{aligned}$$

Using a simple identity  $(m+1)_k/k! = (k+1)_m/m!$ , we get

$$K_{D_{n,m}}((z, \zeta), (z', \zeta')) = \frac{\mu^n}{\pi^{n+m}} e^{\mu m \langle z, z' \rangle} \sum_{k=0}^{\infty} (k+1)_m (k+m)^n e^{\mu k \langle z, z' \rangle} \langle \zeta, \zeta' \rangle^k.$$

Here we remark that

$$|e^{\mu \langle z, z' \rangle} \langle \zeta, \zeta' \rangle| < 1 \quad (5)$$

for all  $(z, \zeta), (z', \zeta') \in D_{n,m}$ . Indeed, from the definition of  $D_{n,m}$  and the Cauchy-Schwartz inequality, we see that  $|\langle \zeta, \zeta' \rangle|^2 \leq \|\zeta\|^2 \|\zeta'\|^2 < e^{-\mu(\|z\|^2 + \|z'\|^2)}$ , for any  $(z, \zeta), (z', \zeta') \in D_{n,m}$ . Combining this and a simple inequality  $\|z\|^2 + \|z'\|^2 \geq 2\operatorname{Re} \langle z, z' \rangle$ , we have  $|\langle \zeta, \zeta' \rangle|^2 < |e^{-\mu \langle z, z' \rangle}|^2$ . Hence  $|e^{\mu \langle z, z' \rangle} \langle \zeta, \zeta' \rangle| < 1$ .

Let us evaluate the series

$$H_{m,n}((z, \zeta), (z', \zeta')) = \sum_{k=0}^{\infty} (k+1)_m (k+m)^n e^{\mu k \langle z, z' \rangle} \langle \zeta, \zeta' \rangle^k. \quad (6)$$

It is easy to see from (1) that the  $m$ -th derivative of the polylogarithm function has the following series representation:

$$\frac{d^m Li_s(z)}{dz^m} = \sum_{k=m}^{\infty} (k-m+1)_m k^{-s} z^{k-m} \quad (7)$$

$$= \sum_{k=0}^{\infty} (k+1)_m (k+m)^{-s} z^k, \quad (8)$$

for  $|z| < 1$ . Comparing (6) and (8), we obtain

$$H_{m,n}((z, \zeta), (z', \zeta')) = \frac{d^m}{dt^m} Li_{-n}(t) \Big|_{t=e^{\mu \langle z, z' \rangle} \langle \zeta, \zeta' \rangle}.$$

This proves the formula (3). The formula (4) follows from (3) and a well-known property of the polylogarithm function [6, eq. 2.1]:

$$\frac{d}{dt} Li_s(t) = \frac{Li_{s-1}(t)}{t}.$$

We have just completed the proof of Theorem 2.  $\square$

**Remark 1.** *There is a following closed form of the  $m$ -th derivative of the polylogarithm function:*

$$\frac{d^m Li_{-n}(t)}{dt^m} = \frac{m! \sum_{j=0}^n (-1)^{n+j} (m+1)_j S(1+n, 1+j) (1-t)^{n-j}}{(1-t)^{n+m+1}}, \quad (9)$$

where  $S(\cdot, \cdot)$  denotes the Stirling number of the second kind (see [6]).

#### 4. AN APPLICATION

As an application of Theorem 2 we solve the Lu Qi-Keng problem for  $D_{n,m}$  in some cases. The Lu Qi-Keng problem asks whether the Bergman kernel has zeros or not. He posed this problem in connection with the global well-definedness of the representative coordinates (see [2]). Lu Qi-Keng's recent result [14] implies that the zero of the Bergman kernel has a geometric interpretation.

We begin with the following lemma which together with the inequality (5) tells us that the image of the map  $D_{n,m} \times D_{n,m} \ni ((z, \zeta), (z', \zeta')) \mapsto e^{\mu\langle z, z' \rangle} \langle \zeta, \zeta' \rangle \in \mathbb{C}$  is the unit disk.

**Lemma 1.** *For any  $\alpha \in \mathbb{C}$  such that  $|\alpha| < 1$ , there exist  $(z, \zeta), (z', \zeta') \in D_{n,m}$  such that  $\alpha = e^{\mu\langle z, z' \rangle} \langle \zeta, \zeta' \rangle$ .*

*Proof.* Let  $\alpha = re^{i\theta}$ ,  $r < 1$ . For any fixed  $z \in \mathbb{C}^n$ , we can choose  $\zeta \in \mathbb{C}^m$  such that  $\|\zeta\|^2 = re^{-\mu\|z\|^2}$ . Then  $(z, \zeta), (z, e^{-i\theta}\zeta) \in D_{n,m}$  and  $e^{\mu\langle z, z \rangle} \langle \zeta, e^{-i\theta}\zeta \rangle = re^{i\theta}$ .  $\square$

We next discuss the location of zeros of the polylogarithm function and its derivative.

**Lemma 2.** *The function  $Li_{-n}(z)/z$  has a zero  $z_0$  such that  $|z_0| < 1$  for all  $n \geq 3$ .*

*Proof.* It is well-known that the Eulerian polynomial has only negative real, simple roots (see [5, p. 292, Exercise 3]). Since  $n \geq 3$ , there exists a root  $\alpha$  such that  $|\alpha| \neq 1$ . If  $|\alpha| < 1$ , then  $\alpha$  is a desired zero. Now we assume that  $|\alpha| > 1$ . Then the following formula [6, eq.(2.2)]

$$Li_{-n}\left(\frac{1}{z}\right) = (-1)^{n+1} Li_{-n}(z) \quad (n \in \mathbb{N}),$$

implies that  $\alpha^{-1}$  is a desired zero.  $\square$

Further information of the location of zeros of  $Li_{-n}(z)/z$  is found in [15]. This short proof was obtained by private communication with Prof. Ochiai and Dr. Shiomi (compare with the proof in [15]).

The following is immediate from a straightforward computation.

**Lemma 3.** *For any  $m \in \mathbb{N}$ , the  $(m-1)$ -th derivative of  $Li_{-2}(t)/t$  is expressed as*

$$\frac{d^{m-1}}{dt^{m-1}} \frac{Li_{-2}(t)}{t} = \frac{(m+1)!(t+m)}{(1-t)^{m+2}}.$$

From this lemma, we see that  $t = -m$  is the zero of  $\frac{d^{m-1}}{dt^{m-1}} \frac{Li_{-2}(t)}{t}$ .

Summarizing, we get:

**Theorem 3.** *The Bergman kernel  $K_{D_{n,m}}$  is zero-free if  $n = 1$  and  $m \geq 1$ . If  $m = 1$  and  $n \geq 2$  then  $K_{D_{n,m}}$  has a zero.*

**Remark 2.** For our domain  $D_{n,m}$ , the solution of the Lu Qi-Keng problem depends only on the value  $(m, n)$ . In general, the solutions of the Lu Qi-Keng problem for the Hartogs domains  $\{(z, \zeta) \in \Omega \times \mathbb{C}^m; \|\zeta\|^2 < p(z)^\mu\}$  depend not only on  $(m, n)$  but also on  $\mu$  (cf. [7]).

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GRADUATE SCHOOL OF MATHEMATICS, NAGOYA UNIVERSITY, FURO-CHO, CHIKUSA-KU, NAGOYA 464-8602, JAPAN

E-mail address: d08006u@math.nagoya-u.ac.jp